Number Shapes

Mathematics is the search for pattern. For children of primary age there are few places where this search can be more satisfyingly pursued than in the field of figurate numbers - numbers represented as geometrical shapes.

Chapter I of these notes shows models the children can build from interlocking cubes and marbles, how they are related and how they appear on the multiplication square.

Chapter II suggests how masterclasses exploiting this material can be organised for children from year 5 to year 9.

Chapter I

Taken together, **Sections 1** (pp. 4-5), **2** (pp. 6-8) and **3** (pp. 9-10) constitute a grand tour. For those involved in initial teacher training or continued professional development, the map of the whole continent appears on p. 3. The 3 sections explore overlapping regions. In each case, there are alternative routes to the final destination - **A** and **B** in the following summary:

Section 1
A) Add a pair of consecutive *natural* numbers and you get an *odd* number; add the consecutive *odd* numbers and you get a *square* number.
B) Add the consecutive *natural* numbers and you get a *triangular* number; add a pair of consecutive *triangular* numbers and you also get a *square* number.

Section 2
A) Add a pair of consecutive *triangular* numbers and you get a *square* number; add the consecutive *square* numbers and you get a *pyramidal* number.
B) Add the consecutive *triangle* numbers and you get a *tetrahedral* number; add a pair of consecutive *tetrahedral* numbers and you also get a *pyramidal* number.

Section 3
(A) Add a pair of consecutive *square* numbers and you get a *centred square* number; add the consecutive *centred square* numbers and you get an *octahedral* number.

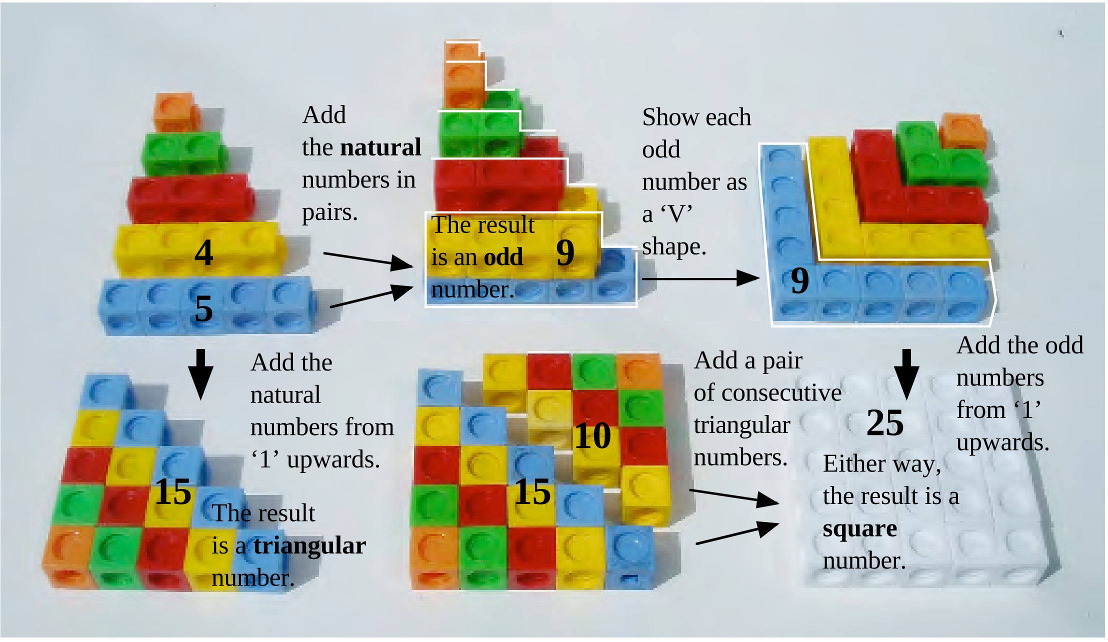
B) Add the consecutive *square* numbers and you get a *pyramidal* number; add a pair of consecutive *pyramidal* numbers and you also get an *octahedral* number.

An addition chart for figurate numbers

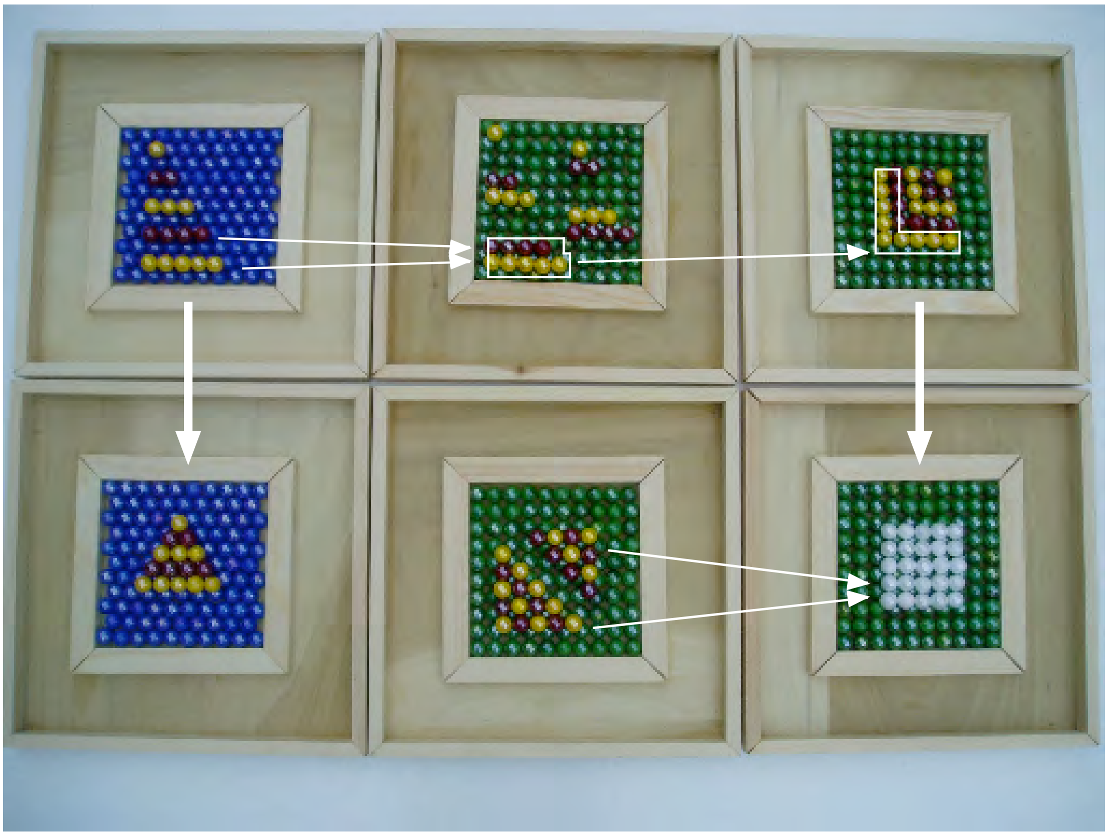
A figurate number is a number shown as a shape.
In the first row of this table the shapes are 1-dimensional (a line, a V-shape).
In the second row they are 2-dimensional (a triangle, a square, a ‘centred’ square).
In the third row they are 3-dimensional (a tetrahedron, a pyramid, an octahedron).

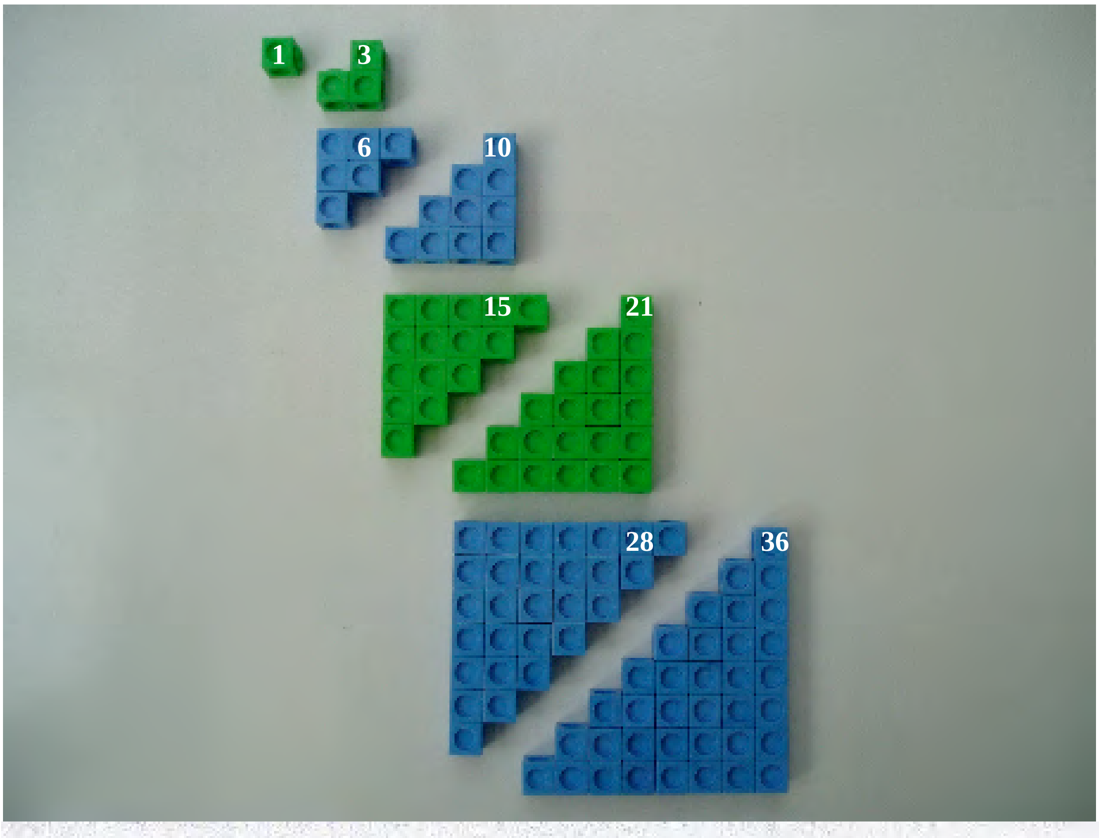
Key: ↓ Add all → Add consecutive pair

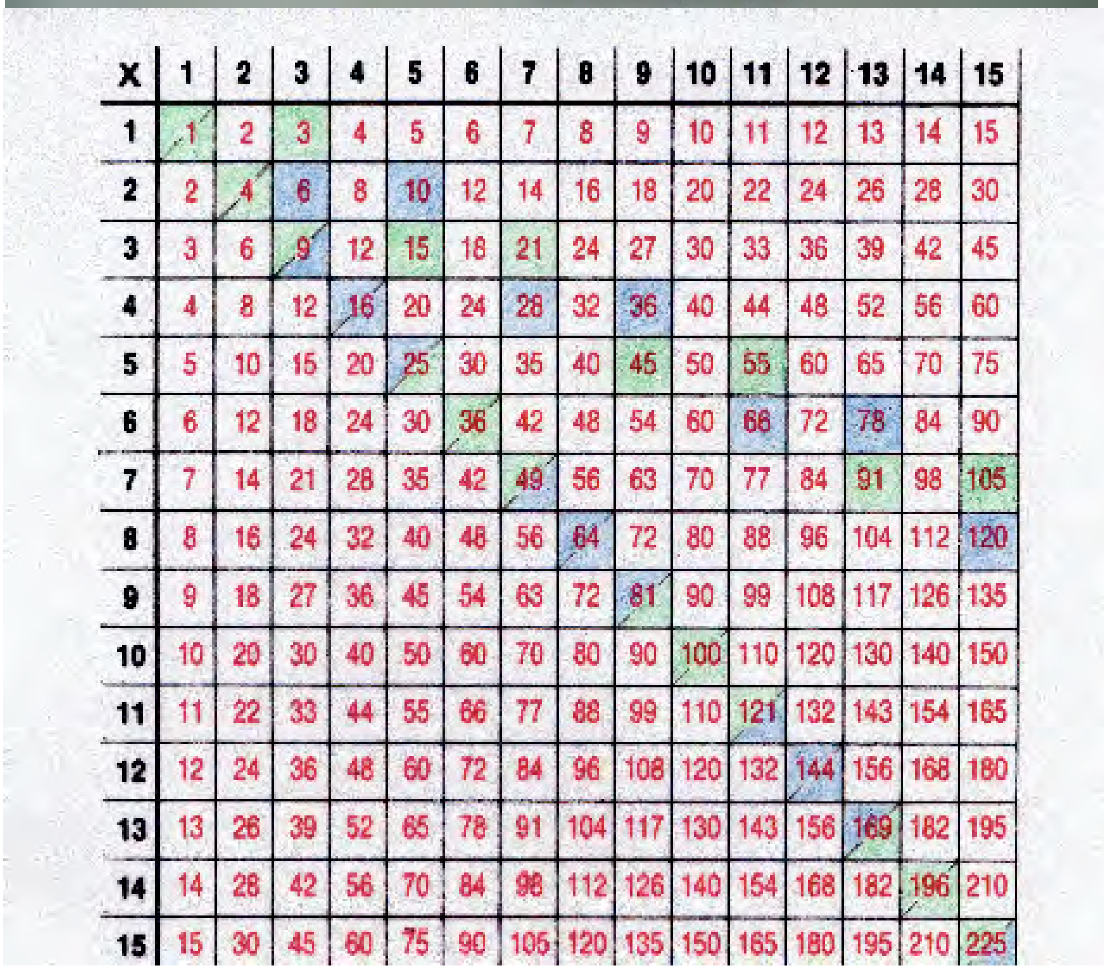
Natural numbers n = 1,2,3, …	Odd numbers 2n – 1 = 1,3,5, …	
Triangular numbers $T_n = \frac{n(n+1)}{2}$ = 1,3,6, …	Square numbers $S_n = T_{n-1} + T_n$ $= n^2$ = 1,4,9, …	Centred square numbers $S_{n-1} + S_n$ $= 4T_{n-1} + 1$ $= 2n^2 - 2n + 1$ = 1,5,13, …
Tetrahedral numbers $Tet_n = \frac{n(n+1)(n+2)}{6}$ = 1,4,10, …	Pyramidal numbers $Pyr_n = Tet_{n-1} + Tet_n = \frac{n(n+1)(2n+1)}{6}$ = 1,5,14, …	Octahedral numbers $Oct_n = Pyr_{n-1} + Pyr_n = Tet_{n-2} + 2Tet_{n-1} + Tet_n = \frac{n(2n^2+1)}{3}$ = 1,6,19, …



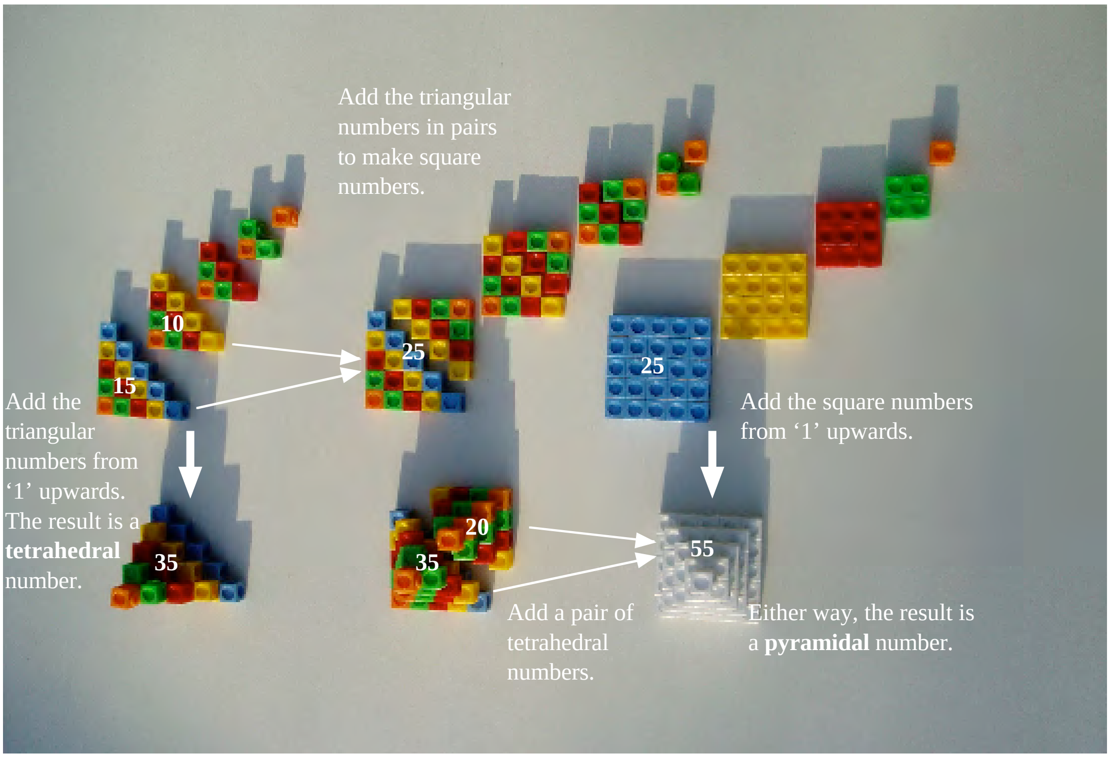
Add
the **natural**
numbers in
pairs.
4
5
The result
is an **odd**
number.
9
Show each
odd
number as
a ‘V’
shape.
9
Add the
natural
numbers from
‘1’ upwards.
15
The result
is a **triangular**
number.
10
15
Add a pair
of consecutive
triangular
numbers.
25
Add the odd
numbers
from ‘1’
upwards.
Either way,
the result is a
square
number.

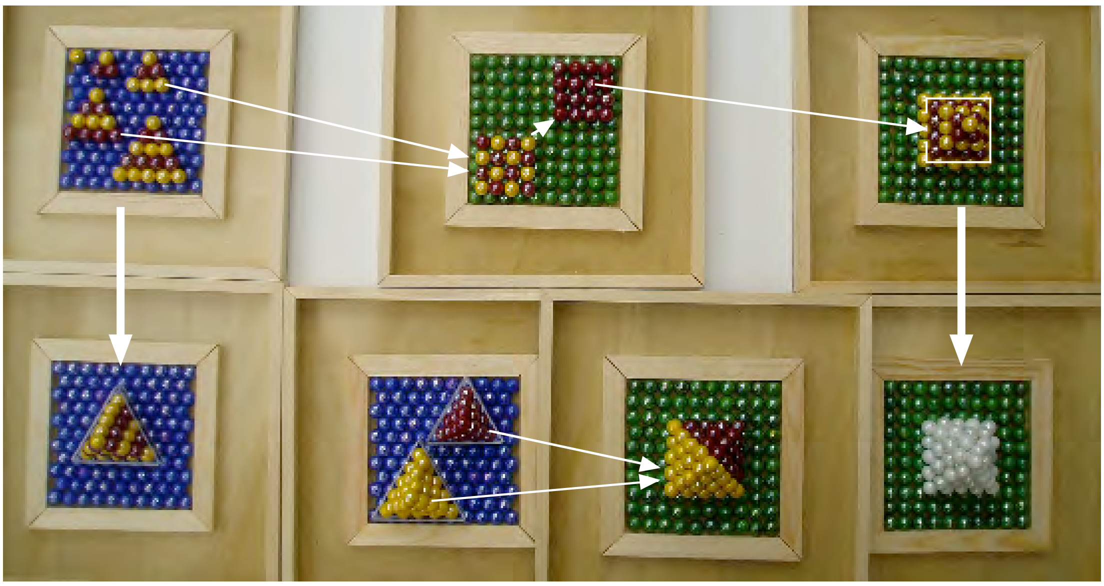


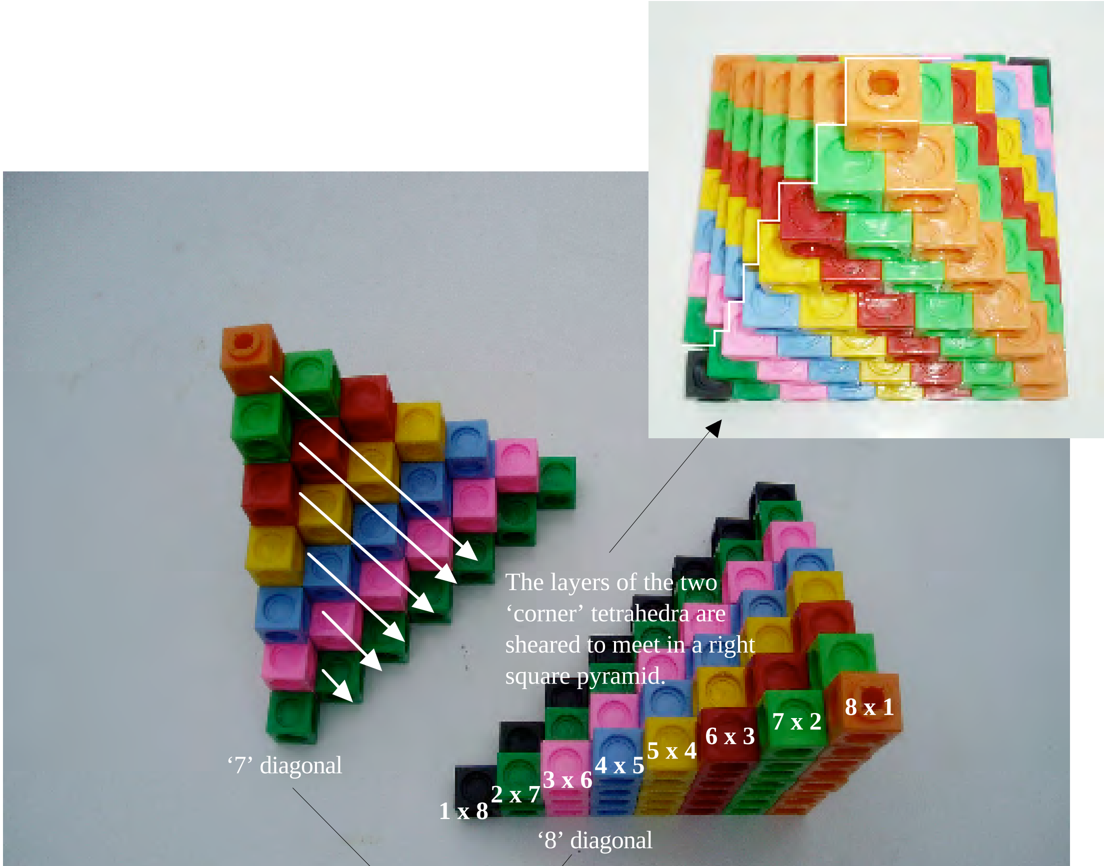




X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225


Add the triangular numbers in pairs to make square numbers.
10
15
25
25
Add the triangular numbers from '1' upwards. The result is a **tetrahedral** number.
35
20
35
55
Add the square numbers from '1' upwards.
Add a pair of tetrahedral numbers.
Either way, the result is a **pyramidal** number.





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2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
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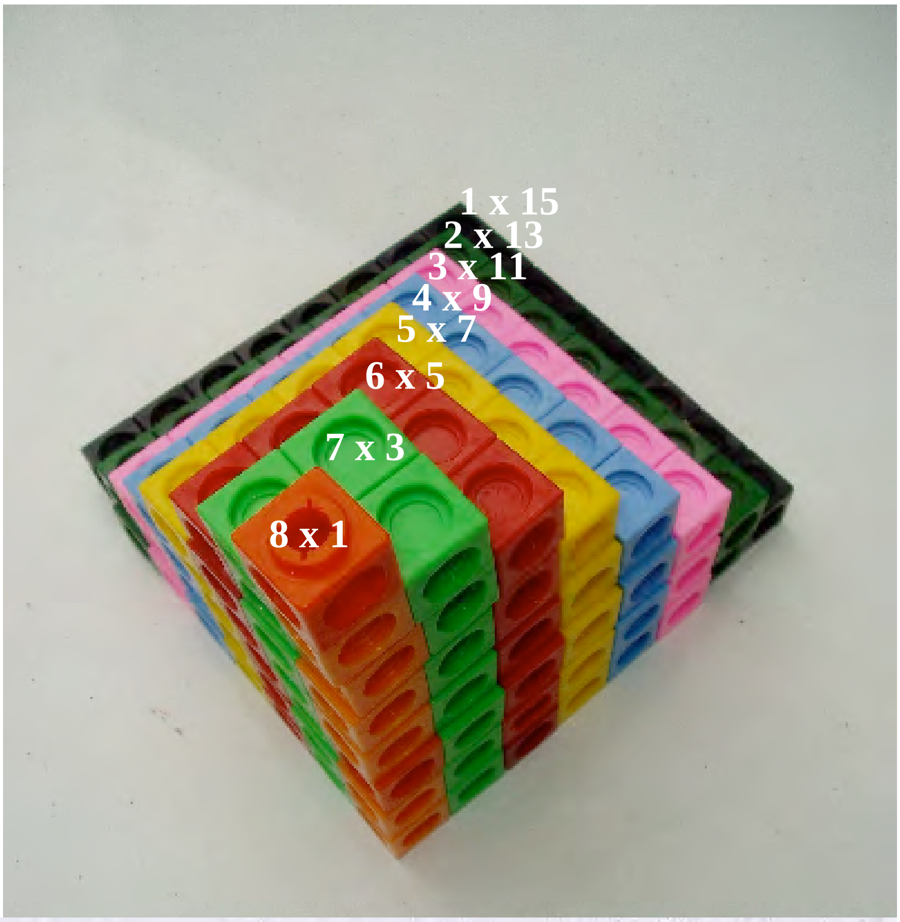

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
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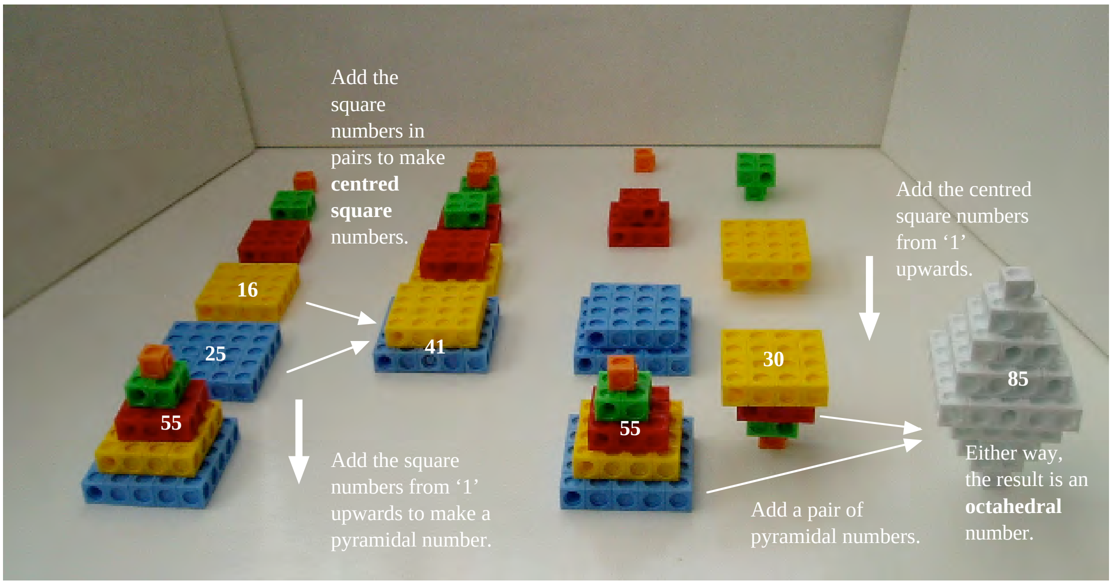
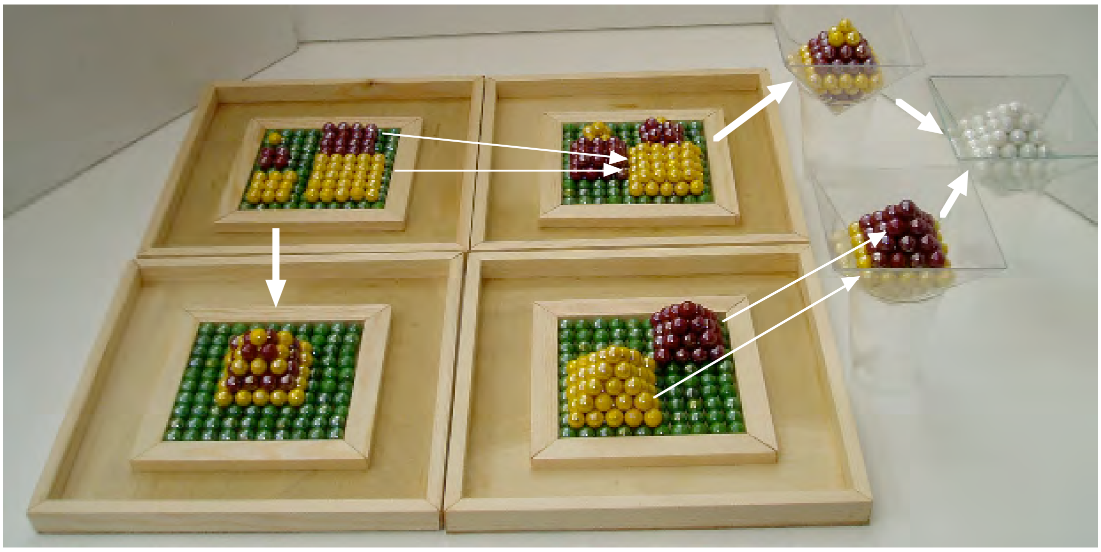
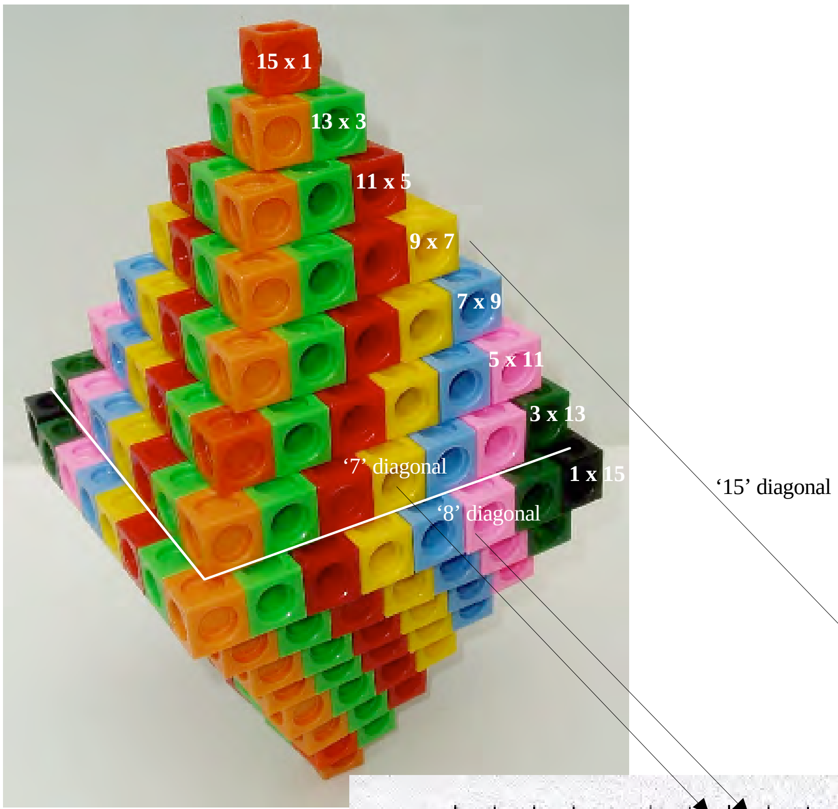

Add the
square
numbers in
pairs to make
**centred
square**
numbers.
Add the centred
square numbers
from '1'
upwards.
16
25
41
30
55
55
85
Add the square
numbers from '1'
upwards to make a
pyramidal number.
Add a pair of
pyramidal numbers.
Either way,
the result is an
octahedral
number.





x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
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Chapter II

For our typical primary child we have in mind an able Y5; for our secondary child, an able Y8.

The essential activities are the experiments (**E**). Those more appropriate to the primary children are tagged '(Y5)'; those more appropriate to the secondary children, '(Y8)'; those – the majority – suited to both ages groups, '(Y5+Y8)'.

The same notation goes for the pieces of mathematics (**M**). The general results described verbally by the younger children are expressed in the language of algebra by the older students.

Schedule

The session falls naturally into the following two parts:
Part 1: 2-D shapes
Part 2: 3-D shapes

Apparatus

The constructions are based on two grids, realised, according to dimension, as follows:

ISOMETRIC	dotty paper	blue marble tray	Multilink sliders in brick wall position	tetrahedral hopper
SQUARE	dotty paper	green marble tray	Multilink sliders arranged as grid	pyramidal hopper

The patterns are made on the dotty paper simply by colouring the dots with crayon. In the marble trays, marbles are stuck down in positions corresponding to those dots but the pattern is made by sitting loose marbles in the depressions formed between 3 of the fixed ones. Marbles of 6 colours are provided.

The Multilink sliders are simply sticks of cubes. By shifting one stick half a cube with respect to an adjacent one, a triangular grid is produced.

Shifting the sticks between the brick wall position and the standard grid is equivalent to moving between the blue and green marble trays.

In 3 dimensions, freestanding forms can be built from Multilink slabs. With marbles, either one stacks the marbles on the trays or does so in inverted form in the hoppers.

The hoppers stand in plastic cups, apex downwards. The tetrahedral one is so designed that it can also be set with a pair of opposite edges horizontal in a special stand (again made from a plastic cup).

The advantage of building the inverted forms is that one assembles the 2-D sections in ascending order of size.

You, the session leader, have an OHP on which you can arrange counters in positions corresponding to those of the marbles.

You will be aware that marbles – particularly of a single colour – must be specially sourced; the trays (illustrated in **Chapter I**) must be specially built, but (as the parallel illustrations in **Chapter I** also make clear), *everything which can be done with marbles can be done with Multilink cubes.*

Though we have in mind a series of activities to be accomplished in a single morning or afternoon, I shall include in my notes several – we can call them 'extension activities' – which really belong in a maths club, that is to say a weekly place and time where students can access a set of apparatus and play with it.

Teaching points

1. The relations between the shapes are paralleled by the relations between the numbers.

2. The numbers which emerge can be found on the multiplication square either as individual cells or as sums of cells arranged along diagonals.

3. The same applies to Pascal's Triangle. The Y8 students will be about to meet this array formally and we can give them a foretaste.

Part 1: 2-D shapes

Which shapes are the most useful to choose?
This first experiment is designed to show that complicated shapes should be regarded as composites of – should be dissected into – a few simple ones.

E1 (Y5): Ask the children to choose the first letter of their name. (We shall take as example ‘L’ and ‘V’.)

L:
Problem? The foot and tail of the ‘L’ could be different lengths.
Solution? Break the ‘L’ into two rectangles.

V:
Problem? The triangle at the bottom could be very big, the parallelogram-shaped sides very small.
Solution: Break the ‘V’ into a triangle and two parallelograms.

The next experiment is designed to show the equivalence of certain of these simple shapes in terms of the numbers they represent.

E2 (Y5): By using the Multilink sliders or changing between the blue and green marble boards the children can convert:
equilateral triangles into right-angled isosceles triangles,
parallelograms into rectangles,
rhombuses into squares.

The natural numbers are of only two kinds: those which can be shown as a rectangle and those which cannot – the primes.

E3 (Y5+Y8): List on the whiteboard half a dozen numbers, one of which is a prime. The children will find they can show all the composite numbers as rectangles – perhaps some in more than one way – but the prime only as a row of unit width.

Point out that a triangle number is the sum of consecutive natural numbers, starting with 1.

M1 (Y8): Show how Pascal’s Triangle is constructed. Invite the children to make their own down to, say, row 8. Show how, as a consequence of the construction method, an entry in one diagonal sums all preceding entries to that point in the next-lower diagonal (the ‘Christmas Stocking’ theorem). Ask the children where the triangle numbers fall.

This activity is not essential to the development but provides a little exercise in pattern recognition and prediction.

E4 (Y5+Y8):

a) Start with the first triangle number and grow further ones by adding borders. What sequence results? (Answer: numbers 1, 4, 7, ... in the sequence. The borders – the first order differences – are the multiples of 9.)

b) What about the borders if you start with the first *square* number and build out from there? (Answer: the multiples of 8.)

c) What about the borders if you start with the first *centred hexagon* number? (Answer: the multiples of 6.)

The pattern is clearer if we write the sequence for **a**, **b**, **c** not as 9,8,6 but as 3x3, 4x2, 6x1.

In the same way that a triangle number is the sum of consecutive natural numbers, a square number is the sum of consecutive odd numbers.

E5 (Y5):

a) Build an odd number as a rectangle of width 2 units with a single unit on the end. Rebuild it by using the single unit as the corner of an ‘L’ and splitting the rectangle to provide an equal foot and tail.

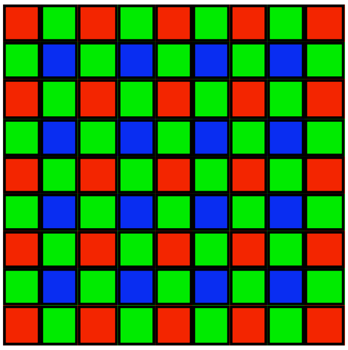
b) Added to a square, this piece completes a larger one - it is the ‘gnomon’ to a square. Grow a square from a single unit by adding successive gnomons – i.e. successive odd numbers.

M2 (Y5): *Where do the square numbers fall on the multiplication square?* (A given entry on the multiplication square lies on a rectangular hyperbola: primes only at the ends, the squares *of* primes in exactly 3 positions, products of two different primes in 4, squares *of* products of two different primes in 9, so the children will find cells in many positions and we must ask a more focused question:) *A square number is a number timesed by itself: where do all square numbers fall on the multiplication square?* (Answer: down the main diagonal. The children should colour them in on a multiplication square provided.)

The next number type lies on the ‘grand tour’ charted in **Chapter I**. It is only the sum of a pair of consecutive squares but has two contrasting figurate disguises.

E6 (Y5+Y8): If you have access to a floor tiled in a checkerboard pattern – for example the ground floor of the home of mathematics masterclasses, the Royal Institution itself! – this activity can be conducted as ‘people maths’ but, working on the table scale, make a 9x9 checkerboard with black and white Multilink cubes with white cubes at the corners.

a) The first arrangement:
Stick a red cube on each white cube in rows and columns 1, 3, 5, 7 and 9.
Stick a blue cube on each white cube in rows and columns 2, 4, 6 and 8.
Stick a green cube on each black cube.



Notice that the blue cubes form the 4th square number; the red cubes, the 5th.

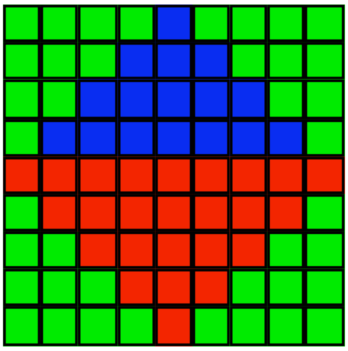
Notice incidentally that the green cubes form a 4x5 rectangle with its long side aligned east-west and an identical rectangle with its long side aligned north-south.

The children should check the arithmetic:

The 'red' total	= 25
The 'blue' total	= 16
The 'white' = 'red' + 'blue' total	= 41
The 'black' = 'green' total = 2 x (4 x 5)	= 40
The grand total	= 81
The whole square = 9 x 9	= 81: check.

b) The second arrangement:

Redistribute the colours like this:



Looking at rows, notice that:
the blue triangle is the sum of the first 4 odd numbers, therefore – as we are expecting - the 4th square,

the red triangle is the sum of the first 5 odd numbers, therefore – as we are expecting – the 5^{th} square.

But notice that the blue and red triangles together form a square set at 45 degrees. This shows the 5th *centred square* number, the sum of the 4^{th} and 5^{th} squares.

If time allows, it's worth working up to the 9x9 square through the 1x1, 3x3, 5x5 and 7x7 sizes.

It's also worth returning to the green part of the second figure with Y8 when we have found a formula for the n^{th} triangular number.

We now establish a relation between square and triangular numbers.

E7 (Y5+Y8): Build two consecutive triangular numbers – preferably in their 'right-angled' form – the same way round. Now give the smaller one a half turn so that it complements the larger one to form a square.

The next experiment leads to the formula for the n^{th} triangular number – however expressed.

E8 (Y5+Y8): Build two identical triangular numbers – again, preferably in their 'right-angled' form – again, the same way round. Performing experiment **E7** yields in this case not a square but a rectangle. *What is special about this rectangle?* (Answer: the longer side is 1 unit greater than the shorter side.)

M3 (Y8): Invite the children to build a sequence of such rectangles and write down an expression for the area of each in terms of square units:
1 x 2, 2 x 3, 3 x 4, …
How do we work out the area of each triangle? (Answer: halve that product.)
Accordingly, the children write expressions for the constituent triangles:
(1 x 2)/2, (2 x 3)/2, (3 x 4)/2, …

Now point out to the children that a formula is just such an expression but containing a letter for which we can substitute any number we like.

We'll use n.

The 7th triangular number (omitting the 'x' sign before the bracket by convention) is $\frac{7(7+1)}{2}$. *What is the* n^{th} *?* (Answer: $\frac{n(n+1)}{2}$.)

M4 (Y8): Invite the children to simplify the numerical expressions for the consecutive triangles so that they're left with a product of two numbers:

1 x 1, 1 x 3, 2 x 3, 2 x 5, 3 x 5, 3 x 7, …

Ask if they notice any pattern. (The first terms form the repeated sequence of natural numbers. The second terms form the sequence of odd numbers, all except the first repeated.)

With that pattern in mind, they should mark the sequence of triangular numbers on the multiplication square. (It forms a staircase.)

Part 2: 3-D shapes

E9 (Y5+Y8): Use Multilink to build a sequence of cubes as stacks of square prisms of unit thickness.

E10 (Y8):

a) Choose the 5th cube and build it as a series of gnomons, i.e. shells completing 3 coincident faces. (They may colour-code the shells.)

b) Use the blue tray to build the first 5 centred hexagon numbers.

Ask the children what they notice and what they conclude. (The gnomons *are* the centred hexagons. Conclusion: the n^{th} cube is the sum of the first n centred hexagon numbers.)

E11 (Y5+Y8):

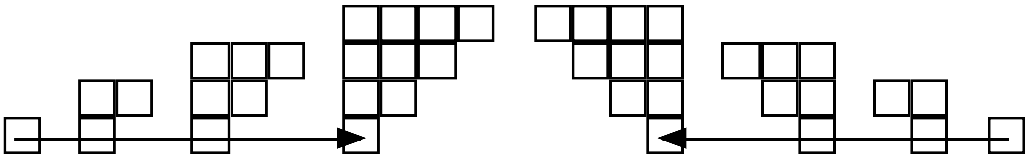
a) In the tetrahedral hopper build the 7th tetrahedral number as a stack of colour-coded triangular layers.

b) With the hopper reorientated in its special stand, build the same number as a stack of colour-coded *rectangular* layers.

M5 (Y5+Y8): Ask the children to read off each rectangle in **E11 (b)** as a product and mark it on a multiplication square. What do they notice? (The products fall on a diagonal.) How then can they work out the 7th tetrahedral number? (Sum the entries down that diagonal.)

M6 (Y8): Ask the children where the 7th tetrahedral number falls on Pascal's Triangle. Point out how the 'Christmas stocking' theorem produces it.

E12 (Y8): Choose the 4th tetrahedral number. Use Multilink to make 6 copies: 3 'left-handed', 3 'right-handed', as shown:



Use a different colour for each set of 3.

The children are then faced with a little dissection puzzle: to fit the 6 pieces together to form a cuboid. They will find the dimensions of the cuboid are 4 x 5 x 6.

Invite the children to count the cubes in the original tetrahedron. (1 + 3 + 6 + 10 = 20.)

How can they use the dimensions of the cuboid to work out the number of cubes in the same piece? (4 x 5 x 6 divided by 6.)

They should now build the sequence of smaller tetrahedral numbers in the same way, look for the number pattern, hazard a formula for the n^{th} tetrahedral number and test it on bigger examples – which they need not build. ($\frac{n(n+1)(n+2)}{6}$.)

E13 (Y5+Y8):

a) In the pyramidal hopper build the 7^{th} pyramidal number as a stack of colour-coded square layers.

b)

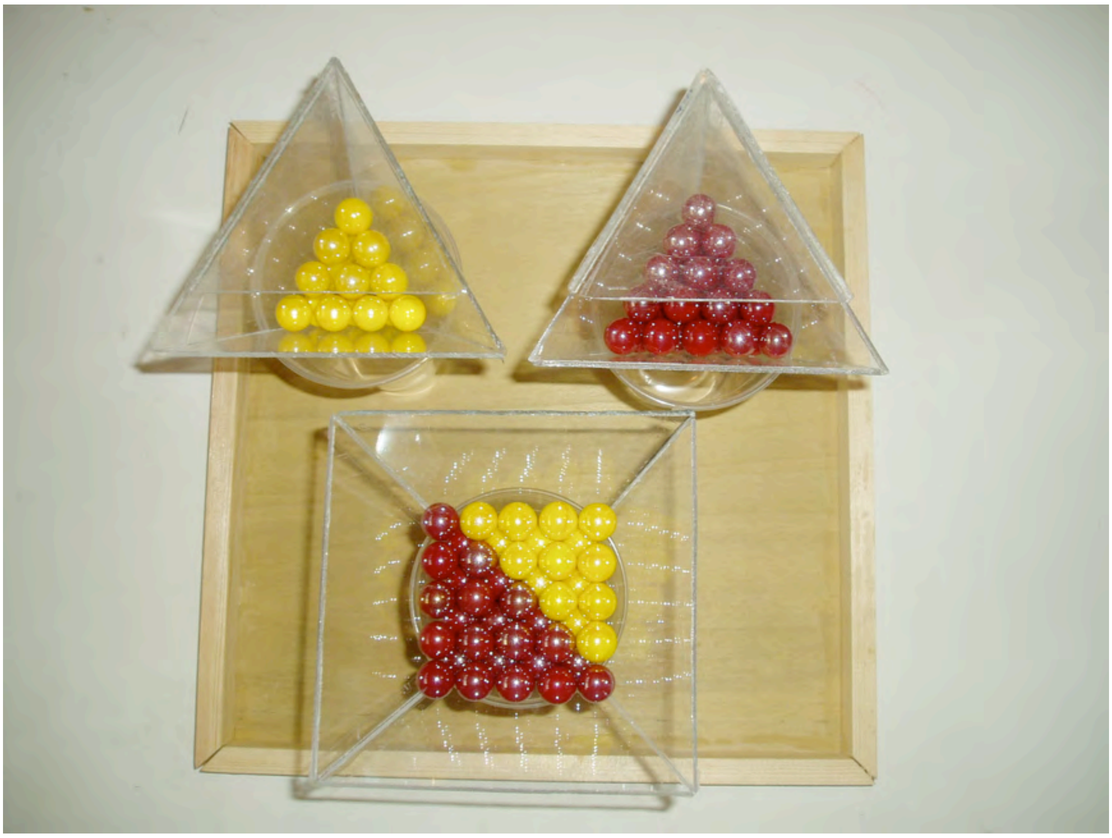
You should now take two tetrahedral hoppers, marbles of two colours (say red and yellow), and demonstrate the second way in which the children are to fill their hoppers:

The right-hand hopper receives 1 red marble; the left-hand, none.
Accordingly the children place 1 red marble in their pyramidal hopper.

The right-hand hopper receives 3 red marbles; the left-hand, 1 yellow marble.
Accordingly the children build a second layer by arranging 3 red marbles in the form of a right-angled triangle and complement this with the 1 yellow marble.

Continue in this way.

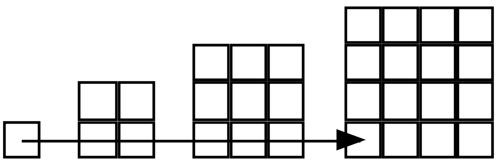
Ask the children what the experiment demonstrates. (They will express themselves in different ways. In ‘Y8’ terms: as the n^{th} square is the sum of the $(n-1)^{th}$ and n^{th} triangles, the n^{th} pyramid is the sum of the $(n-1)^{th}$ and n^{th} tetrahedra.)



M7 (Y8): The children can shade two adjacent diagonals on their multiplication square to show the two tetrahedra whose sum is a pyramid. But ask them to add vertically and mark the cell in each column where the sum falls. They will produce another diagonal but a steeper one (as illustrated in **Chapter I**).

Here we establish the formula for the n^{th} pyramidal number as we did for the n^{th} tetrahedron: by use of a dissection.

E14 (Y8): Use Multilink to build 6 copies of the 4th pyramid, each in a different colour, like this:



As in **E12** the challenge is to assemble 6 pieces into a cuboid. They will find the dimensions this time are 4 x 5 x 9.

The counting methods described in **E12** yield the 4th pyramidal number in these two ways:
1 + 4 + 9 + 16,
4 x 5 x 9 divided by 6.

To get some idea of where the '9' comes from, we recall the finding of **E13** and write down what the expression would be for the sum of the two tetrahedra making up the pyramid:

$$\frac{(3)(4)(5)}{6}+\frac{(4)(5)(6)}{6}=\frac{(4)(5)(3+6)}{6}=\frac{(4)(5)(9)}{6}.$$

The '9' retains its disguise until we generalise the result by means of algebra:

$$\frac{(n-1)n(n+1)}{6}+\frac{n(n+1)(n+2)}{6}=\frac{n(n+1)\left[(n-1)+(n+2)\right]}{6}=\frac{n(n+1)(2n+1)}{6}.$$

You will need to talk the children through this algebra.

The children should add a 5^{th} square prism to each of their units, predict the dimensions of the new cuboid and work out what the 5^{th} pyramidal number is by the second method. They can then check by the first. As in **E12** they can then do the same for bigger pyramids without any model to hand.

E15 (Y5+Y8): Fill the pyramidal hopper as in **E13** but then use the top layer of marbles as a base on which to build an upright pyramid. The result is a double pyramid, an octahedron.

The children have built the 7^{th} octahedral number.

Point out the plane of symmetry running through the middle layer. Ask the Y5 children how this helps counting the marbles. (They can add the squares above the central layer, double their sum and add in the 7^{th} square.)

Point out to the Y8 children that the octahedron is made up of the 6^{th} pyramid joined to the 7^{th} and ask them for an 'instant' total.

M8 (Y8): The process echoes **M7**. But this time, they add the two, steep diagonals *horizontally*. They find they're back to one inclined at 45°, but with the entries spaced out (as shown in **Chapter 1**).

M9 (Y8): As one can add two consecutive tetrahedral numbers to make a pyramid, one can add two consecutive pyramidal numbers to make an octahedron. The corresponding algebra needs a little more manipulation but keen students may like to attempt it:

$$\frac{(n-1)n(2n-1)}{6}+\frac{n(n+1)(2n+1)}{6}=\frac{n\left[(n-1)(2n-1)+(n+1)(2n+1)\right]}{6}$$

$$=\frac{n\left(4n^2+2\right)}{6}=\frac{n\left(2n^2+1\right)}{3}.$$

They should check their formula for a few simple cases – including the one they've just built.

Paul Stephenson, 21.10.09